APPROXIMATION OF SYMMETRIZATIONS BY MARKOV PROCESSES

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Abstract. Under continuity and recurrence assumptions, we prove that the iteration of successive partial symmetrizations that form a time-homogeneous Markov process, converges to a symmetrization. We cover several settings, including the approximation of the spherical nonincreasing rearrangement by Steiner symmetrizations, polarizations and cap symmetrizations. A key tool in our analysis is a quantitative measure of the asymmetry.

1. Introduction and main results

1.1. Approximation by Steiner symmetrizations. Steiner symmetrizations are measure-preserving transformations of sets that bring symmetry with respect to one direction $u \in \mathbf{P}^{d-1}_{\mathbf{R}}$ in the Euclidean space [1]. The resulting set X^u is symmetric with respect to the direction u. It was observed in the study of the classical isoperimetric inequality that any Borel measurable set $X \subseteq \mathbf{R}^d$ which is left invariant under all Steiner symmetrizations must be an Euclidean ball centered on the origin [17].

A natural question is whether the spherical nonincreasing rearrangement, which associates to each Borel measurable set X the unique Euclidean ball $X^\star$ centered on 0 and with the same measure as X, can be approximated by Steiner symmetrizations, that is whether there exists a sequence $(u_n)_{n\in\mathbf{N}}$ such that the sequence of successive Steiner symmetrizations $(X^{u_1\cdots u_n})_{n\in\mathbb{N}}$ converges somehow to the spherical nonincreasing rearrangement $X^\star$. Such results have been obtained in order to prove various properties of symmetrizations [5; 11, proof of theorem 3.7]. The approximation procedure seems quite robust, and this brings the question whether *random sequences* of partial symmetrizations approximate symmetrizations.

Independent random symmetrizations of sets and functions were studied in various settings [7, 8, 12, 20, 23], and rates of convergence were recently discovered [7, Corollary 5.4, Proposition 6.2; 8, Theorem 3]. A typical result for the convergence of independent Steiner symmetrizations is:

Theorem 1.1. *Let $(S_n)_{n\in\mathbf{N}}$ be a sequence of independent and μ-identically distributed sequence of Steiner symmetrizations. We have*

$$\mu(\{u \in \mathbf{P}^{d-1}_{\mathbf{R}} : m(X\Delta X^u) > 0\}) > 0, \tag{1}$$

for every Lebesgue measurable set $X \subseteq \mathbf{R}^d$ of finite measure with $m(X\Delta X^\star) > 0$, if and only if, for every Lebesgue measurable set $X \subseteq \mathbf{R}^d$ of finite measure, the sequence of successive Steiner symmetrizations $(X^{S_1\cdots S_n})_{n\in\mathbf{N}}$ converges almost-surely in measure to $X^\star$.

In this paper we investigate the approximation by *time-homogeneous Markov processes.* A stochastic process $(S_n)_{n\in\mathbf{N}}$ valued in a topological space $\mathscr{S}$ is a time-homogeneous Markov process if there exists a transition function

$$P : \mathscr{S} \times \mathscr{B}(\mathscr{S}) \to [0,1] : (s, A) \mapsto P_s(A),$$

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satisfying some measurability conditions (see [13] or section 2.4 in this text), and such that if, for every $n \in \mathbf{N}$, and for every Borel measurable set $A \in \mathscr{B}(\mathscr{S})$, we have almost-surely

$$\mathbb{P}\{S_{n+1} \in A \big| S_1, \dots, S_n\} = P_{S_n}(A).$$

The iterated kernels P^k are then defined to satisfy ([13] or section 2.4 in this text)

$$P^k_{S_n}(A_1 \times \cdots \times A_k) = \mathbb{P}\{S_{n+1} \in A_1, \dots, S_{n+k} \in A_k \big| S_1, \dots, S_n\},$$

almost-surely for all $n, k \in \mathbf{N}$ and all Borel measurable sets $A_1, \dots, A_k \in \mathscr{B}(\mathscr{S})$. In contrast with processes made up of independent and identically distributed variables, successive Steiner symmetrizations that form a Markov process are mutually correlated. In the deterministic case, such a dependence can be an obstruction to convergence [3, 7, 24]. We obtain the following result for Steiner symmetrizations:

Theorem 1.2. *Let $(S_n)_{n\in\mathbf{N}}$ be a time-homogeneous Markov process of Steiner symmetrizations with initial distribution μ. If there exists $\mathfrak{s}_\star \in \mathbf{P}^{d-1}_{\mathbf{R}}$ such that*

(i) (Recurrence) for every nonempty open set $\mathfrak{s}_\star \in \mathcal{O} \subseteq \mathbf{P}^{d-1}_{\mathbf{R}}$, we have

$$\mathbb{P}((S_n)_{n\in\mathbf{N}} \textit{ enters } \mathcal{O} \textit{ infinitely many often}) = 1,$$

(ii) (Continuity) for every $n \in \mathbf{N}$ and for every open set $\mathcal{O} \subseteq (\mathbf{P}^{d-1}_{\mathbf{R}})^n$, the map

$$s \in \mathbf{P}^{d-1}_{\mathbf{R}} \mapsto P^n_s(\mathcal{O})$$

is lower semi-continuous at $\mathfrak{s}_\star$,

(iii) (Discrimination) for every Lebesgue measurable set $X \subseteq \mathbf{R}^d$ of finite measure, with $m(X \Delta X^\star) > 0$, the process of Steiner symmetrizations starting at $\mathfrak{s}_\star$ reaches in finite time the set $\{u \in \mathbf{P}^{d-1}_{\mathbf{R}} : m(X \Delta X^u) > 0\}$, that is:

$$\sum_{n\in\mathbf{N}} P^n_{\mathfrak{s}_\star}(\,(\mathbf{P}^{d-1}_{\mathbf{R}})^{n-1} \times \{u \in \mathbf{P}^{d-1}_{\mathbf{R}} : m(X \Delta X^u) > 0\}\,) > 0;$$

then for every Lebesgue measurable set $X \subseteq \mathbf{R}^d$ of finite measure, the sequence $(X^{S_1 \dots S_n})_{n\in\mathbf{N}}$ converges in measure to $X^\star$, almost-surely.

The discrimination condition (iii) in theorem 1.2 is similar to condition (1) in theorem 1.1, and they are equivalent for independent sequences of random symmetrizations. A necessary condition for the conclusion of theorem 1.2 to hold is that for each Lebesgue measurable set X of finite measure and with $m(X \Delta X^\star) > 0$, we should have

$$\sum_{n\in\mathbf{N}} \mathbb{P}(\{m(X \Delta X^{S_n}) > 0\}) > 0. \tag{2}$$

By Fubini's theorem, for each X, condition (2) implies that condition (iii) holds for some $s \in \mathbf{P}^{d-1}_{\mathbf{R}}$, but s may depends on X and n. Therefore, condition (2) is close, but not equivalent, to condition (iii) in theorem 1.2. The initial distribution μ of the process is only involved in the recurrence condition (i): the recurrent point $\mathfrak{s}_\star$ is assumed to be deterministic, so that its existence does not simply follows from the compactness of the projective plane $\mathbf{P}^{d-1}_{\mathbf{R}}$. The continuity condition in theorem 1.2 is stronger than the usual weak-Feller continuity at $\mathfrak{s}_\star$, but still weaker than the usual strong Feller-continuity everywhere (see also [13] for definitions, and proposition 2.14 and discussion below for details). The recurrence condition (i) and continuity condition (ii) ensure together that the asymptotic behaviour of the process is closed to an independent process with distribution $P_{\mathfrak{s}_\star}$.

In the proof of theorems 1.1 and 1.2, we do not study directly the distance between sets, in order to prove the convergence. We rather measure the convergence with an asymmetry, which is a functional of the form

$$\mathcal{A}(X) = \int_X \frac{|x|^2}{1 + |x|^2}\,\mathrm{d}x. \tag{3}$$

The asymmetry function strictly decreases along Steiner symmetrizations of X, and reaches a minimum at $X^\star$. The idea of using such a function to measure the asymmetry of sets is a standard technique in the field of symmetrizations (see for example [2, 7, 22, 23]).

1.2. **Other symmetrizations.** The spherical nonincreasing rearrangement has been approximated by other partial symmetrizations, such as cap symmetrizations [15] and polarizations [1, 6, 20, 21, 24]. Other symmetrizations such as the cap symmetrization [16] and discrete symmetrizations [14] have also been approximated in the deterministic case in order to prove isoperimetric theorems.

Theorem 1.2 *fails* for the approximation of the spherical nonincreasing rearrangement by successive cap symmetrizations or polarizations. Polarizations and cap symmetrizations can be thought of as elements in $S^{d-1}\times[0,+\infty)$ (see also section 3.1 for accurate definitions). In fact, any cap symmetrization is characterized by a affine half line passing through the origin $0\in\mathbf{R}^d$, which yields $S^{d-1}\times[0,+\infty)$ as parametrization space. Although cap symmetrizations from $S^{d-1}\times(0,+\infty)$ – those cap symmetrizations whose half line has edge different of 0 – strictly decrease the asymmetry (3) of non spherical sets, cap symmetrizations from $S^{d-1}\times\{0\}$ may act as isometries. Such symmetrization could conspire to a non convergent behaviour. On the other hand, they could also be necessary in the realization of the process (see section 3.4 for more details).

In order to prevent bad behaviour without just throwing away cap symmetrizations from $S^{d-1}\times\{0\}$, we strengthen the discrimination condition (iii) from theorem 1.2. This leads us to the following theorem:

Theorem 1.3. *Let $(S_n)_{n\in\mathbf{N}}$ be a time-homogeneous Markov process of cap symmetrizations (resp. polarizations) with initial distribution μ. If there exists $\mathfrak{s}_\star\in S^{d-1}\times[0,+\infty)$ such that*

(i) (Recurrence) for every nonempty open set $\mathfrak{s}_\star\in\mathcal{O}\subseteq S^{d-1}\times[0,+\infty)$, we have

$$\mathbb{P}((S_n)_{n\in\mathbf{N}} \text{ enters } \mathcal{O} \text{ infinitely many often}) = 1,$$

(ii) (Continuity) for every $n\in\mathbf{N}$ and for every open set $\mathcal{O}\subseteq(S^{d-1}\times[0,+\infty))^n$, the map

$$s\in\mathbf{P}^{d-1}_{\mathbf{R}}\mapsto P^n_{\mathfrak{s}_\star}(\mathcal{O})$$

is lower semi-continuous at $\mathfrak{s}_\star$,

(iii) (Discrimination) for every Lebesgue measurable set $X\subseteq\mathbf{R}^d$ of finite measure, we have

$$\sum_{n\in\mathbf{N}} P^n_{\mathfrak{s}_\star}((\,S^{d-1}\times(0,+\infty)\,)^{n-1}\times\{u\in S^{d-1}\times[0,+\infty): m(X\Delta X^u)>0\})>0,$$

then for every Lebesgue measurable set $X\subseteq\mathbf{R}^d$ of finite measure, the sequence $(X^{S_1\cdots S_n})_{n\in\mathbf{N}}$ converges in measure to $X^\star$, almost-surely.

While conditions (i) and (ii) in theorem 1.2 and theorem 1.3 are similar, the discrimination condition (iii) takes into account the bad symmetrizations from $S^{d-1}\times\{0\}$. Condition (iii) means that the process starting at $\mathfrak{s}_\star$ reaches in finite time, any set of the form $\{u\in\mathbf{P}^{d-1}_{\mathbf{R}}: m(X\Delta X^u)>0\}$ *without passing through* $S^{d-1}\times\{0\}$. In the case of independent and identically distributed cap symmetrizations and polarizations, condition (1) tells us that the boundary set $S^{d-1}\times\{0\}$ is not allowed to support the measure μ; thus the discrimination condition (iii) in theorem 1.3 reduces to (1).

1.3. **Organization of the paper.** In order to emphasize the main properties of symmetrizations that we use, we work in section 2 with an abstract notion of symmetrizations that covers Steiner and cap symmetrizations, and polarizations. We draw the reader attention to the fact that the abstract framework we work with, is mainly aimed to strip the proofs of non pertinent particularities. Without any assumptions on the process $(S_n)_{n\in\mathbf{N}}$, not even the Markov property, we prove an abstract criterion to test the convergence, proposition 2.7 (section 2.2). The strategy here is a summability trick used by Burchard and Fortier [7]. This abstract

criterion is then particularized in two directions. We first deduce an abstract version of 1.1. A second particularization is a general result about Markov processes, from which we derive theorems 1.2 and 1.3. We finally give several more explicit examples and results. A particular attention is drawn to a *new example* for cap symmetrizations, where the boundary $S^{d-1} \times \{0\}$ is needed for the universal convergence and, thus, can not be simply removed from the parameter space for the cap symmetrizations and polarizations.

2. Abstract convergence result

2.1. **Abstract symmetrizations.** We fix a metric space $(\mathscr{X}, d)$ and a nonexpansive projector $\star$ in $(\mathscr{X}, d)$, that is a map $[X \in \mathscr{X} \mapsto X^\star \in \mathscr{X}]$ such that, for all $X, Y \in \mathscr{X}$, we have $d(X^\star, Y^\star) \leq d(X, Y)$ and $X^{\star\star} = X^\star$. We introduce the following abstract setting for symmetrizations.

Definition 2.1. *A* symmetrization space *is a nonempty set $\mathscr{S}$ of maps $[X \in \mathscr{X} \mapsto X^s \in \mathscr{X}]$ endowed with a metrizable topology with countable basis, such that*

(a) (Continuity) the map $[(X, s) \in \mathscr{X} \times \mathscr{S} \mapsto X^s]$ is continuous,
(b) (Idempotence) for every $X \in \mathscr{X}$, $X^{ss} = X^s$,
(c) (Nonexpansiveness) for all $X, Y \in \mathscr{X}$, $d(X^s, Y^s) \leq d(X, Y)$.

The elements of $\mathscr{X}$ are called objects*, and elements of $\mathscr{S}$ are called* symmetrizations*.*

In view of the nonexpansiveness (c), the continuity (a) can be deduced from the apparently weaker assumption that, for every $X \in \mathscr{X}$, the map $[s \in \mathscr{S} \mapsto X^s]$ is continuous.

Definition 2.2. *We say that a symmetrization space $\mathscr{S}$ is* $\star$-compatible *if*

(a) for every $s \in \mathscr{S}$, for every $X \in \mathscr{X}$, we have $X^{s\star} = X^\star = X^{\star s}$,
(b) for every $X \in \mathscr{X}$, if $X = X^s$ for every $s \in \mathscr{S}$, then $X = X^\star$.

Definition 2.3. *Let $\mathscr{S}$ be a symmetrization space. A function $\mathcal{A} : \mathscr{X} \to \mathbf{R}$ is an* asymmetry *on $\mathscr{S}$ if $\mathcal{A}$ is continuous and if for every $s \in \mathscr{S}$, for every $X \in \mathscr{X}$, we have $\mathcal{A}(X^s) \leq \mathcal{A}(X)$. An asymmetry function $\mathcal{A}$ is said to be a* strict asymmetry *on $\mathscr{S}$ when for every $X \in \mathscr{X}$, for every $s \in \mathscr{S}$, the equality $\mathcal{A}(X^s) = \mathcal{A}(X)$ implies $X^s = X$.*

Definition 2.4. *Let $\mathscr{S}$ be a symmetrization space and $\mathcal{A}$ be an asymmetry function on $\mathscr{S}$. We say that $\mathcal{A}$ is* $\star$-compatible *if for every $X \in \mathscr{X}$ satisfying $\mathcal{A}(X) \leq \mathcal{A}(X^\star)$, we have $X = X^\star$.*

If a symmetrization space $\mathscr{S}$ on $(\mathscr{X}, d)$ is $\star$-compatible, then the function

$$\mathcal{A} : \mathscr{X} \to \mathbf{R} : \mathcal{A}(X) = d(X, X^\star)$$

always defines a $\star$-compatible asymmetry function on $\mathscr{S}$. However, this asymmetry function might not be the best choice in convergence theory. Another direct consequence of the definitions is that every strict asymmetry function on a $\star$-compatible symmetrization space $\mathscr{S}$, is itself $\star$-compatible.

The next proposition characterizes the convergence in $\mathscr{X}$ of iterated symmetrizations in terms of the asymmetry.

Proposition 2.5. *Let $\mathscr{S}$ be a symmetrization space, $\mathcal{A}$ be an asymmetry function on $\mathscr{S}$, $(s_n)_{n \in \mathbf{N}}$ be a sequence in $\mathscr{S}$ and $X \in \mathscr{X}$. If $\mathscr{S}$ and $\mathcal{A}$ are $\star$-compatible, then*

$$\liminf_{n \to +\infty} d(X^\star, X^{s_1 \cdots s_n}) = 0$$

if and only if, the set $\{X^{s_1 \cdots s_n} : n \in \mathbf{N}\}$ has compact closure in $\mathscr{X}$ and

$$\liminf_{n \to +\infty} \mathcal{A}(X^{s_1 \cdots s_n}) = \mathcal{A}(X^\star).$$

Proof. The "only if" part is a consequence of the continuity of $\mathcal{A}$, and the fact that the closure of a convergence sequence is always compact. For the converse, assume that the set $\{X^{s_1\cdots s_n} : n \in \mathbf{N}\}$ has compact closure, and that

$$\liminf_{n\to+\infty} \mathcal{A}(X^{s_1\cdots s_n}) = \mathcal{A}(X^\star).$$

Since the asymmetry function decreases along symmetrizations, the sequence $(\mathcal{A}(X^{s_1\cdots s_n}))_{n\in\mathbf{N}}$ converges to $\mathcal{A}(X^\star)$. By compactness assumption, there exists a subsequence $(X^{s_1\cdots s_{n_k}})_{k\in\mathbf{N}}$ that converges to some $Y \in \mathscr{X}$. By continuity of $\mathcal{A}$, we have

$$\mathcal{A}(X^\star) = \lim_{k\to+\infty} \mathcal{A}(X^{s_1\cdots s_{n_k}}) = \mathcal{A}(Y).$$

The $\star$-compatibility of $\mathscr{S}$ then implies

$$d(X^\star, Y^\star) = \lim_{k\to+\infty} d(X^\star, X^{s_1\cdots s_{n_k}\star}) = d(X^\star, X^\star) = 0,$$

so that $X^\star = Y^\star$ and $\mathcal{A}(Y^\star) = \mathcal{A}(Y)$. Since $\mathcal{A}$ is $\star$-compatible, we deduce that $Y = X^\star$. Since this is true for each accumulation point of the sequence $(X^{s_1\cdots s_n})_{n\in\mathbf{N}}$, by compactness, this sequence converges to $X^\star$. □

2.2. **Abstract result for random symmetrizations.** From now on, we fix a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ and $(S_n)_{n\in\mathbf{N}}$ a sequence of measurable maps from $(\Omega, \mathscr{A})$ to $\mathscr{S}$, which is assumed to be a symmetrization space, endowed with its Borel σ-algebra $\mathscr{B}(\mathscr{S})$. For every $n \in \mathbf{N}$, we write $\mathcal{F}_n$ the sub-σ-algebra of $\mathscr{A}$ induced by $\{S_1, \ldots, S_n\}$, and $\mathcal{F}$ the smallest sub-σ-algebra of $\mathscr{A}$ that contains $\bigcup_{n\in\mathbf{N}} \mathcal{F}_n$. If N is a stopping time adapted to $(S_n)_{n\in\mathbf{N}}$, its induced filtration is denoted by $\mathcal{F}_N$. Throughout the text, we write $\mathbb{P}(\cdot)$ (resp. $\mathbb{E}$) probabilities (resp. expectations), and $\mathbb{E}\{\cdot|\cdot\}$ conditional expectations.

The next technical lemma allows to reduce the randomness by taking the infimum; it follows from the classical properties of conditional expectation.

Lemma 2.6. *Let $\mathscr{S}$ be a symmetrization space and $\mathcal{B} \subseteq \mathcal{F}$ be a σ-algebra. Let be $X \in \mathscr{X}$, and set*

$$X^{\mathscr{S}} = \{X^{s_1\cdots s_n} : n \in \mathbf{N}, s_1, \ldots, s_n \in \mathscr{S}\}.$$

If $f : X^{\mathscr{S}} \times \mathscr{S} \to \mathbf{R}$ is continuous and bounded, and if $\mathfrak{G} : (\Omega, \mathcal{B}) \to X^{\mathscr{S}}$ and $S : (\Omega, \mathcal{F}) \to \mathscr{S}$ are measurable, then we have for every $U \in \mathscr{X}$, almost-surely on $\mathfrak{G}^{-1}(U)$,

$$\mathbb{E}\{f(\mathfrak{G}, S)\big|\mathcal{B}\} \geq \inf_{Y\in U} \mathbb{E}\{f(Y, S)\big|\mathcal{B}\}.$$

Proof. Without loss of generality, we can assume that f is positive. The topological space $X^{\mathscr{S}} \times \mathscr{S}$ is second-countable. Therefore it is not difficult to prove that, since f is bounded and continuous on $X^{\mathscr{S}} \times \mathscr{S}$, for all probability measure μ on $\mathscr{B}(X^{\mathscr{S}} \times \mathscr{S})$, there exists a decreasing sequence $(f_n)_{n\in\mathbf{N}}$ of simple functions, converging μ-almost-everywhere to f, and whose level sets are finite unions of disjoint Borel rectangles.

We apply this approximation scheme with the conjoint distribution μ of the random vector $(\mathfrak{G}, S)$. Let $(f_n)_{n\in\mathbf{N}}$ be the corresponding approximation sequence. According to standard properties of the conditional expectation [4, Theorem 34.3], we have for every $n \in \mathbf{N}$, almost-surely on $\mathfrak{G}^{-1}(U)$,

$$\mathbb{E}\{f_n(\mathfrak{G}, S)\big|\mathcal{B}\} \geq \inf_{Y\in U} \mathbb{E}\{f_n(Y, S)\big|\mathcal{B}\}.$$

It now follows from the monotone convergence theorem for the conditional expectation that, almost-surely on $\mathfrak{G}^{-1}(U)$,

$$\begin{aligned}\mathbb{E}\{f(\mathfrak{G}, S)\big|\mathcal{B}\} = \inf_{n\in\mathbf{N}} \mathbb{E}\{f_n(\mathfrak{G}, S)\big|\mathcal{B}\} &\geq \inf_{n\in\mathbf{N}} \inf_{Y\in U} \mathbb{E}\{f_n(Y, S)\big|\mathcal{B}\} \\ &\geq \inf_{n\in\mathbf{N}} \inf_{Y\in U} \mathbb{E}\{f(Y, S)\big|\mathcal{B}\} = \inf_{Y\in U} \mathbb{E}\{f(Y, S)\big|\mathcal{B}\}.\end{aligned}$$ □

We are now ready to prove the main result about general stochastic processes of symmetrizations, which need not be Markov processes.

Proposition 2.7 (Convergence by divergence)**.** *Let $\mathscr{S}$ be a symmetrization space and $\mathcal{A}$ be an asymmetry function on $\mathscr{S}$, such that $\mathscr{S}$ and $\mathcal{A}$ are both $\star$-compatible. Let be $X \in \mathscr{X}$. If the set*

$$X^{\mathscr{S}} = \{X^{s_1 \dots s_n} : n \in \mathbf{N}, s_1, \dots, s_n \in \mathscr{S}\}$$

has compact closure in $(\mathscr{X}, d)$, and if there exists an increasing and almost-surely finite sequence of stopping times $(N_n)_{n \in \mathbf{N}}$ adapted to $(S_n)_{n \in \mathbf{N}}$, such that we have almost-surely for all $\epsilon > 0$

$$\sum_{n \in \mathbf{N}} \inf \Big\{ \mathbb{E}\{\mathcal{A}(Y) - \mathcal{A}(Y^{s_{N_n+1} \dots s_{N_{(n+1)}}}) \big| \mathcal{F}_{N_n}\} \ : Y \in \overline{X^{\mathscr{S}}},\ \mathcal{A}(Y) \geq \mathcal{A}(X^\star) + \varepsilon \Big\} = +\infty, \tag{4}$$

then the sequence $(X^{S_1 \dots S_n})_{n \in \mathbf{N}}$ converges almost-surely to $X^\star$.

The proof is based on a summability trick found by Burchard and Fortier [7].

Proof. If $X = X^\star$, there is nothing to prove. Otherwise, by $\star$-compatibility of $\mathcal{A}$, there exists $\varepsilon > 0$ such that $\mathcal{A}(X) \geq \mathcal{A}(X^\star) + \varepsilon$. For $m \in \mathbf{N}$, we write almost-surely

$$\begin{aligned} \mathcal{A}(X) - \mathcal{A}(X^\star) &= \mathcal{A}(X) - \mathcal{A}(X^{S_1 \dots S_{N_1}}) + \mathcal{A}(X^{S_1 \dots S_{N_{(m+1)}}}) - \mathcal{A}(X^\star) \\ &\quad + \sum_{n=1}^{m} \mathcal{A}(X^{S_1 \dots S_{N_n}}) - \mathcal{A}(X^{S_1 \dots S_{N_{(n+1)}}}) \\ &\geq \sum_{n=1}^{m} \mathcal{A}(X^{S_1 \dots S_{N_n}}) - \mathcal{A}(X^{S_1 \dots S_{N_{(n+1)}}}) \\ &\geq \sum_{n=1}^{m} \chi_{\Theta_n} \cdot (\mathcal{A}(X^{S_1 \dots S_{N_n}}) - \mathcal{A}(X^{S_1 \dots S_{N_{(n+1)}}})). \end{aligned}$$

where, for every $n \in \mathbf{N}$, we have defined the set

$$\Theta_n^\varepsilon = \{\mathcal{A}(X^{S_1 \dots S_{N_n}}) \geq \mathcal{A}(X^\star) + \varepsilon\} \in \mathcal{F}_{N_n}.$$

(Here the symbol χ denotes indicator functions of sets.) Taking the expectation on both sides, we compute

$$\begin{aligned} \mathcal{A}(X) - \mathcal{A}(X^\star) &\geq \sum_{n=1}^{m} \mathbb{E}(\chi_{\Theta_n^\varepsilon} \cdot (\mathcal{A}(X^{S_1 \dots S_{N_n}}) - \mathcal{A}(X^{S_1 \dots S_{N_{(n+1)}}}))) \\ &= \sum_{n=1}^{m} \mathbb{E}(\chi_{\Theta_n^\varepsilon} \mathbb{E}\{\mathcal{A}(X^{S_1 \dots S_{N_n}}) - \mathcal{A}(X^{S_1 \dots S_{N_{(n+1)}}}) \big| \mathcal{F}_{N_n}\}). \end{aligned}$$

According to lemma 2.6, and writing $\overline{X_\varepsilon^{\mathscr{S}}} = \{Y \in \overline{X^{\mathscr{S}}} : \mathcal{A}(Y) \geq \mathcal{A}(X^\star) + \varepsilon\}$, we get

$$\begin{aligned} \mathcal{A}(X) - \mathcal{A}(X^\star) &\geq \sum_{n=1}^{m} \mathbb{E}(\chi_{\Theta_n^\varepsilon} \inf_{Y \in \overline{X_\varepsilon^{\mathscr{S}}}} \mathbb{E}\{\mathcal{A}(Y) - \mathcal{A}(Y^{S_{N_n+1} \dots S_{N_{(n+1)}}}) \big| \mathcal{F}_{N_n}\}) \\ &= \mathbb{E}\left(\sum_{n=1}^{m} \chi_{\Theta_n^\varepsilon} \inf_{Y \in \overline{X_\varepsilon^{\mathscr{S}}}} \mathbb{E}\{\mathcal{A}(Y) - \mathcal{A}(Y^{S_{N_n+1} \dots S_{N_{(n+1)}}}) \big| \mathcal{F}_{N_n}\} \right). \end{aligned}$$

Letting $m \in \mathbf{N}$ tend to $+\infty$, the monotone convergence theorem ensures

$$\mathcal{A}(X) - \mathcal{A}(X^\star) \geq \mathbb{E}\left(\sum_{n \in \mathbf{N}} \chi_{\Theta_n^\varepsilon} \inf_{Y \in \overline{X_\varepsilon^{\mathscr{S}}}} \mathbb{E}\{\mathcal{A}(Y) - \mathcal{A}(Y^{S_{N_n+1} \dots S_{N_{(n+1)}}}) \big| \mathcal{F}_{N_n}\} \right).$$

Therefore, we have almost-surely

$$\sum_{n\in\mathbf{N}} \chi_{\Theta_n^\varepsilon} \inf_{Y\in\overline{X_\varepsilon^{\mathscr{S}}}} \mathbb{E}\{\mathcal{A}(Y) - \mathcal{A}(Y^{S_{N_n+1}\dots S_{N_{(n+1)}}})\big|\mathcal{F}_{N_n}\} < +\infty.$$

According to identity (4) and the monotonicity of $\mathcal{A}$ along symmetrizations, the sequence $(\chi_{\Theta_n^\varepsilon})_{n\in\mathbf{N}}$ reaches 0 almost-surely after finitely many steps, so that we have almost-surely

$$\lim_{n\to\infty} \mathcal{A}(X^{S_1\dots S_n}) \le \mathcal{A}(X^\star) + \varepsilon.$$

Now, considering only rational $\varepsilon > 0$, we deduce that the sequence $(\mathcal{A}(X^{S_1\dots S_n}))_{n\in\mathbf{N}}$ converges almost-surely to $\mathcal{A}(X^\star)$. By proposition 2.5, the sequence $(X^{S_1\dots S_n})_{n\in\mathbf{N}}$ converges almost-surely to $X^\star$. $\square$

2.3. **Convergence of independent processes.** In the context of processes made up from independent and identically distributed symmetrizations, we obtain an abstract result from which theorem 1.1 directly follows.

Theorem 2.8. *Let $\mathscr{S}$ be a $\star$-compatible symmetrization space. Let $(S_n)_{n\in\mathbf{N}}$ be a sequence of independent and μ-identically distributed variables. If, for every $X\in\mathscr{X}$, the set*

$$X^{\mathscr{S}} = \{X^{s_1\dots s_n} : n\in\mathbf{N}, s_1,\dots,s_n\in\mathscr{S}\}$$

has compact closure in $(\mathscr{X}, d)$, then the following conditions are equivalent:
- *(i) for every $X\in\mathscr{X}$, the sequence $(X^{S_1\dots S_n})_{n\in\mathbf{N}}$ converges almost-surely to $X^\star$,*
- *(ii) for every $X\in\mathscr{X}$ with $X\ne X^\star$, we have $\mu(\{s\in\mathscr{S} : X\ne X^s\}) > 0$.*

Proof. Let us first assume that, for every $X\in\mathscr{X}$ with $X\ne X^\star$, we have

$$\mu(\{s\in\mathscr{S} : X\ne X^s\}) > 0.$$

We apply proposition 2.7 with the $\star$-compatible asymmetry

$$\mathcal{A} : \mathscr{X} \to \mathbf{R} : \mathcal{A}(X) = d(X, X^\star).$$

We consider the increasing sequence of deterministic stopping times $(N_n)_{n\in\mathbf{N}} = (n)_{n\in\mathbf{N}}$. Let be $X\in\mathscr{X}$ with $d(X,X^\star)\ge\varepsilon>0$ (otherwise there is nothing to prove) and set

$$\overline{X_\varepsilon^{\mathscr{S}}} = \{Y\in\overline{X^{\mathscr{S}}} : d(Y,X^\star)\ge\varepsilon\}.$$

Proposition 2.7 shows that it is sufficient to prove

$$\inf_{Y\in\overline{X_\varepsilon^{\mathscr{S}}}} \int_{\mathscr{S}} d(Y,Y^\star) - d(Y^s,Y^\star)\mu(\mathrm{d}s) > 0.$$

By continuity of $\mathcal{A}$ and compactness of $\overline{X^{\mathscr{S}}}$, it suffices to prove that for all $Y\in\overline{X_\varepsilon^{\mathscr{S}}}$, we have

$$\int_{\mathscr{S}} d(Y,Y^\star) - d(Y^s,Y^\star)\mu(\mathrm{d}s) > 0.$$

Let us fix $Y\in\overline{X_\varepsilon^{\mathscr{S}}}$. By assumption, there exists some $\delta>0$ such that

$$\mu(\{s\in\mathscr{S} : d(Y,Y^\star)\ge\delta\}) > 0.$$

Therefore, we have

$$\int_{\mathscr{S}} d(Y,Y^\star) - d(Y^s,Y^\star)\mu(\mathrm{d}s) \ge \delta\mu(\{s\in\mathscr{S} : d(Y,Y^\star)\ge\delta\}) > 0.$$

This concludes the first part of the alternative.

For the converse, assume that for every $X\in\mathscr{X}$, the sequence $(X^{S_1\dots S_n})_{n\in\mathbf{N}}$ converges almost-surely to $X^\star$. Assume by contradiction that there exists $X\in\mathscr{X}$, $X\ne X^\star$, such that

$$\mu(\{s\in\mathscr{S} : X\ne X^s\}) = 0$$

or, equivalently,

$$\mathbb{P}(\{\omega \in \Omega : X = X^{S_1(\omega)}\}) = 1.$$

Then the sequence $(X^{S_1 \dots S_n})_{n \in \mathbf{N}}$ is almost-surely constant and equals X, hence $X = X^\star$ by assumption on $(S_n)_{n\in\mathbf{N}}$. □

2.4. **Markov processes.** In order to deal with Markov processes, we recall some classical terminology about transition functions. In a metrizable topological space $\mathscr{S}$ with countable basis, a transition function on $\mathscr{S}$ is a function

$$P : \mathscr{S} \times \mathscr{B}(\mathscr{S}) \to [0,1] : (s, A) \mapsto P_s(A)$$

such that

(a) for every $s \in \mathscr{S}$, the function

$$P_s : \mathscr{B}(\mathscr{S}) \to [0,1] : A \mapsto P_s(A)$$

is a probability measure,

(b) for every Borel measurable and bounded function $f : \mathscr{S} \to \mathbf{R}$, the function

$$Pf : \mathscr{S} \to [0,1] : s \mapsto P_s f = \int_{\mathscr{S}} f(y)\, P_s(\mathrm{d}y)$$

is (bounded and) Borel measurable.

For every $n \in \mathbf{N}$ and every rectangle $A_1 \times \dots \times A_n \in \mathscr{B}(\mathscr{S})^n$, the iterated kernel P^n is

$$P^n(A_1 \times \dots \times A_n) : \mathscr{S} \to \mathbf{R} : s \mapsto \int_{\mathscr{S}} \dots \int_{\mathscr{S}} \Big(\prod_{i=1}^{n} \chi_{A_i}(s_i)\Big) P_{s_{n-1}}(\mathrm{d}s_n) \cdots P_{s_1}(\mathrm{d}s_2)\, P_s(\mathrm{d}s_1) \tag{5}$$

We recall [13, Chapter 3] that the stochastic process $(S_n)_{n\in\mathbf{N}}$ is a time-homogeneous Markov process on $\mathscr{S}$ if there exists a transition function P on $\mathscr{S}$ such that for all $n, k \in \mathbf{N}$ with $k > 1$, for every Borel set $A_1 \times \dots \times A_k \subseteq \mathscr{S}^k$, we have (almost-surely)

$$\mathbb{E}\{\chi_{A_1 \times \dots \times A_k}(S_{n+1}, \dots, S_{n+k}) \big| \mathcal{F}_n\} = P^n_{S_n}(A_1 \times \dots \times A_k). \tag{6}$$

Since we deal with discrete time processes, this equality extends to stopping times. Furthermore, identities (5) and (6) directly extend to bounded and continuous functions $f : \mathscr{S}^n \to \mathbf{R}$, following the same approximation scheme of lemma 2.6.

Proposition 2.9. *Let $\mathscr{S}$ be a $\star$-compatible symmetrization space such that for every $X \in \mathscr{X}$, the set*

$$X^{\mathscr{S}} = \{X^{s_1 \dots s_n} : n \in \mathbf{N}, s_1, \dots, s_n \in \mathscr{S}\}$$

has compact closure in $(\mathscr{X}, d)$. Let $\mathcal{A}$ be a $\star$-compatible asymmetry function on $\mathscr{S}$. Assume that there exists $\mathcal{I}^{(\mathscr{S})} \in \mathscr{B}(\mathscr{S})$ such that $\mathcal{A}$ is strict on $\mathscr{S} \setminus \mathcal{I}^{(\mathscr{S})}$. If there exists $\mathfrak{s}_\star \in \mathscr{S}$ such that

(i) (Recurrence) for every nonempty open set $\mathfrak{s}_\star \in \mathcal{O} \subseteq \mathscr{S}$, we have

$$\mathbb{P}((S_n)_{n\in\mathbf{N}} \text{ enters } \mathcal{O} \text{ infinitely many often}) = 1,$$

(ii) (Continuity) for every $n \in \mathbf{N}$, for every bounded and continuous function $f : \mathscr{S}^n \to \mathbf{R}$, the function $P^n f$ is continuous at $\mathfrak{s}_\star$,

(iii) (Discrimination) for every $X \in \mathscr{X}$ with $X \neq X^\star$, we have

$$\sum_{n\in\mathbf{N}} P^n_{\mathfrak{s}_\star}((\mathscr{S} \setminus \mathcal{I}^{(\mathscr{S})})^{n-1} \times \{s \in \mathscr{S} : X \neq X^s\}) > 0,$$

then for every $X \in \mathscr{X}$, the sequence $(X^{S_1 \dots S_n})_{n\in\mathbf{N}}$ converges almost-surely to $X^\star$.

Proof. We apply proposition 2.7. Let us consider a sequence $(\mathcal{O}_n)_{n\in\mathbf{N}}$ of nonempty open sets in $\mathscr{S}$ decreasing to the limit point $\mathfrak{s}_\star$, that is

$$\bigcap_{n\in\mathbf{N}} \mathcal{O}_n = \{\mathfrak{s}_\star\}.$$

We define the stopping time $N_1 = \min\{k \in \mathbf{N} : S_k \in \mathcal{O}_1\}$, and for every $n \in \mathbf{N}$,

$$N_{n+1} = \min\{k \in \mathbf{N} : k \geq N_n + n, S_k \in \mathcal{O}_{n+1}\}.$$

By recurrence assumption (i), the sequence $(N_n)_{n\in\mathbf{N}}$ is a sequence of stopping times which is almost-surely finite and satisfies almost-surely $S_{N_n} \in \mathcal{O}_n$ and $N_{n+1} - N_n \geq n$. Fix $\omega \in \Omega$ such that the previous relations hold, and write $(s_n)_{n\in\mathbf{N}} = (S_{N_n}(\omega))_{n\in\mathbf{N}}$ and $\ell_n = N_{n+1}(\omega) - N_n(\omega)$ for every $n \in \mathbf{N}$. The sequence $(s_n)_{n\in\mathbf{N}}$ converges to $\mathfrak{s}_\star$. Fix $X \in \mathscr{X}$ with $\mathcal{A}(X) \geq \mathcal{A}(X^\star) + \varepsilon$ (otherwise there is nothing to prove). By proposition 2.7, we only need to show

$$\sum_{n\in\mathbf{N}} \inf\Big\{ \int_{\mathscr{S}^{\ell_n}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1\ldots u_{\ell_n}})\, P_{s_n}^{\ell_n}(\mathrm{d}u_1,\ldots,\mathrm{d}u_{\ell_n}) : \\ Y \in \overline{X^{\mathscr{S}}},\, \mathcal{A}(Y) \geq \mathcal{A}(X^\star) + \varepsilon \Big\} = +\infty, \tag{7}$$

By compactness of $\overline{X_\varepsilon^{\mathscr{S}}}$, for every $n \in \mathbf{N}$, there exists $Y_n \in \overline{X_\varepsilon^{\mathscr{S}}}$ such that

$$\inf_{Y\in\overline{X_\varepsilon^{\mathscr{S}}}} \int_{\mathscr{S}^{\ell_n}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1\ldots u_{\ell_n}})\, P_{s_n}^{\ell_n}(\mathrm{d}u_1,\ldots,\mathrm{d}u_{\ell_n}) \\ = \int_{\mathscr{S}^{\ell_n}} \mathcal{A}(Y_n) - \mathcal{A}(Y_n^{u_1\ldots u_{\ell_n}})\, P_{s_n}^{\ell_n}(\mathrm{d}u_1,\ldots,\mathrm{d}u_{\ell_n})$$

Since $\overline{X_\varepsilon^{\mathscr{S}}}$ is compact, there exists a subsequence $(Y_{n_k})_{k\in\mathbf{N}}$ that converges to some $Y \in \overline{X_\varepsilon^{\mathscr{S}}}$. Without loss of generality, we can assume

$$\int_{\mathscr{S}^{\ell_{n_k}}} \mathcal{A}(Y_{n_k}) - \mathcal{A}(Y_{n_k}^{u_1\ldots u_{\ell_{n_k}}})\, P_{s_{n_k}}^{\ell_{n_k}}(\mathrm{d}u_1,\ldots,\mathrm{d}u_{\ell_{n_k}}) \\ \geq \int_{\mathscr{S}^{\ell_{n_k}}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1\ldots u_{\ell_{n_k}}})\, P_{s_{n_k}}^{\ell_{n_k}}(\mathrm{d}u_1,\ldots,\mathrm{d}u_{\ell_{n_k}}) - \frac{1}{2^k}.$$

It is now sufficient to check that

$$\sum_{k\in\mathbf{N}} \int_{\mathscr{S}^{\ell_{n_k}}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1\ldots u_{\ell_{n_k}}})\, P_{s_{n_k}}^{\ell_{n_k}}(\mathrm{d}u_1,\ldots,\mathrm{d}u_{\ell_{n_k}}) = +\infty.$$

By the continuity assumption (ii), for all $k \in \mathbf{N}$, there exists a smaller integer $j_k \in \mathbf{N}$ such that for all $j \geq j_k$,

$$\Big| \int_{\mathscr{S}^{\ell_{n_k}}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1\ldots u_{\ell_{n_k}}})\, P_{s_j}^{\ell_{n_k}}(\mathrm{d}u_1,\ldots,\mathrm{d}u_{\ell_{n_k}}) \\ - \int_{\mathscr{S}^{\ell_{n_k}}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1\ldots u_{\ell_{n_k}}})\, P_{\mathfrak{s}_\star}^{\ell_{n_k}}(\mathrm{d}u_1,\ldots,\mathrm{d}u_{\ell_{n_k}}) \Big| \leq \frac{1}{2^k}.$$

Define $m_1 = \min\{n_k : n_k \geq j_1\}$, and by recurrence

$$m_{k+1} = \min\{n_k : n_k \geq \max\{m_k + 1, j_{k+1}, n_{(k+1)}\}\}.$$

By construction, $(m_k)_{k\in\mathbf{N}}$ is a subsequence of $(n_k)_{n\in\mathbf{N}}$ such that $m_k \ge j_k$ for all $k \in \mathbf{N}$. Since the asymmetry decreases along symmetrizations, we have for every $k \in \mathbf{N}$

$$\begin{aligned}
\int\limits_{\mathscr{S}^{\ell_{m_k}}} &\mathcal{A}(Y) - \mathcal{A}(Y^{u_1\dots u_{\ell_{m_k}}})\, P^{\ell_{m_k}}_{s_{m_k}}(\mathrm{d}u_1,\dots,\mathrm{d}u_{\ell_{m_k}})\\
&\ge \int\limits_{\mathscr{S}^{\ell_{n_k}}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1\dots u_{\ell_{n_k}}})\, P^{\ell_{n_k}}_{s_{m_k}}(\mathrm{d}u_1,\dots,\mathrm{d}u_{\ell_{n_k}})\\
&\ge \int\limits_{\mathscr{S}^{\ell_{n_k}}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1\dots u_{\ell_{n_k}}})\, P^{\ell_{n_k}}_{\mathfrak{s}_\star}(\mathrm{d}u_1,\dots,\mathrm{d}u_{\ell_{n_k}}) - \frac{1}{2^k}\\
&\ge \int\limits_{\mathscr{S}^{k}} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1\dots u_k})\, P^{k}_{\mathfrak{s}_\star}(\mathrm{d}u_1,\dots,\mathrm{d}u_k) - \frac{1}{2^k}.
\end{aligned}$$

In the previous line, we have used the fact that $\ell_{n_k} \ge n_k \ge k$ by construction. If we prove the strict inequality

$$\sup_{n\in\mathbf{N}} \int\limits_{\mathscr{S}^n} \mathcal{A}(Y) - \mathcal{A}(Y^{u_1\dots u_n})\, P^n_{\mathfrak{s}_\star}(\mathrm{d}u_1,\dots,\mathrm{d}u_n) > 0, \tag{8}$$

then by comparison of series, condition (7) would then hold. Let us thus prove (8), where $Y \in \mathscr{X}$ satisfies $Y \ne Y^\star$. By the discrimination assumption (iii), there exists $n \in \mathbf{N}$ such that

$$P^n_{\mathfrak{s}_\star}((\mathscr{S}\setminus\mathcal{I}^{(\mathscr{S})})^{n-1} \times \{s \in \mathscr{S}\setminus\mathcal{I}^{(\mathscr{S})} : Y \ne Y^s\}) > 0.$$

Since $\mathcal{A}$ is strict on $\mathscr{S}\setminus\mathcal{I}^{(\mathscr{S})}$, there exists $\delta > 0$ such that $P^n_{\mathfrak{s}_\star}(H) > 0$, with

$$H = (\mathscr{S}\setminus\mathcal{I}^{(\mathscr{S})})^{n-1} \times \{s \in \mathscr{S} : \mathcal{A}(Y) \ge \mathcal{A}(Y^s) + \delta\}.$$

Let us assume by contradiction that we have

$$\int\limits_{\mathscr{S}^n} \mathcal{A}(Y) - \mathcal{A}(Y^{s_1\dots s_n})\, P^n_{\mathfrak{s}_\star}(\mathrm{d}s_1,\dots,\mathrm{d}s_n) = 0.$$

There exists a set $E \in \mathscr{B}(\mathscr{S}^n)$ of $P^n_{\mathfrak{s}_\star}$-measure 1 such that, for every $(s_1,\dots,s_n) \in E$, we have $\mathcal{A}(Y) = \mathcal{A}(Y^{s_1\dots s_n})$. For $(s_1,\dots,s_n) \in E$, with $s_1 \notin \mathcal{I}^{(\mathscr{S})}$, we have $\mathcal{A}(Y) = \mathcal{A}(Y^{s_1})$ and, since $\mathcal{A}$ is a strict asymmetry function on $\mathscr{S}\setminus\mathcal{I}^{(\mathscr{S})}$, we have $Y = Y^{s_1}$, so that we also have $\mathcal{A}(Y) = \mathcal{A}(Y^{s_2})$. By recurrence, we have $Y = Y^{s_i}$, $i = 1,\dots,n-1$, for every $(s_1,\dots,s_n) \in E$ with $s_i \notin \mathcal{I}^{(\mathscr{S})}$ for each $i \in \{1,\dots,n-1\}$. We now have

$$\begin{aligned}
0 &= \int\limits_{\mathscr{S}^n} \mathcal{A}(Y) - \mathcal{A}(Y^{s_1\dots s_n})\, P^n_{\mathfrak{s}_\star}(\mathrm{d}s_1,\dots,\mathrm{d}s_n)\\
&\ge \int\limits_{H\cap E} \mathcal{A}(Y) - \mathcal{A}(Y^{s_n})\, P^n_{\mathfrak{s}_\star}(\mathrm{d}s_1,\dots,\mathrm{d}s_n) \ge \delta P^n_{\mathfrak{s}_\star}(H) > 0,
\end{aligned}$$

which is the desired contradiction. This proves

$$\sup_{n\in\mathbf{N}} \int\limits_{\mathscr{S}^n} \mathcal{A}(Y) - \mathcal{A}(Y^{s_1\dots s_n})\, P^n_{\mathfrak{s}_\star}(\mathrm{d}s_1,\dots,\mathrm{d}s_n) > 0,$$

and (8) holds. □

In practice, the proof of the existence of a recurrent point $\mathfrak{s}_\star$ requires some additional work. We recall a notion of stability that is useful for locally compact symmetrization spaces.

Definition 2.10 ([13, Section 9.2]). *Let $\mathscr{S}$ be a locally compact metrizable topological space with countable basis, and $(S_n)_{n\in\mathbf{N}}$ a Markov process on $\mathscr{S}$. The process $(S_n)_{n\in\mathbf{N}}$ is said to be* nonevanescent *if*

$$\mathbb{P}(\exists K \subseteq \mathscr{S} : K \textit{ compact}, (S_n)_{n\in\mathbf{N}} \textit{ enters } K \textit{ infinitely many often}) = 1.$$

The topological assumptions on $\mathscr{S}$ ensure that the above definition makes sense.

Proposition 2.11 ([13, Theorem 9.1.3]). *Let $\mathscr{S}$ be a metrizable topological space with countable basis, and $(S_n)_{n\in\mathbf{N}}$ be a Markov process on $\mathscr{S}$. For every $A \in \mathscr{B}(\mathscr{S})$, the sequence $(\inf_{k\in\mathbf{N}} P^k_{S_n}(\,(\mathscr{S}\setminus A)^k))_{n\in\mathbf{N}}$ converges almost-surely to the indicator function of the set*

$$\bigcup_{n\in\mathbf{N}}\bigcap_{k\geq n}\{S_{k+1}\notin A\}.$$

Proof. We fix a Borel measurable set $A \in \mathscr{B}(\mathscr{S})$, and we consider the function

$$H : \mathscr{S} \mapsto [0,1] : H(s) = 1 - \inf_{k\in\mathbf{N}} P^k_s(\,(\mathscr{S}\setminus A)^k\,).$$

We write, for every $n \in \mathbf{N}$,

$$\Omega_n = \bigcup_{i\geq n}\{S_{i+1}\in A\} \in \mathcal{F},$$

and

$$\Omega_\infty = \bigcap_{m\in\mathbf{N}}\bigcup_{i\geq m}\{S_{i+1}\in A\}.$$

For every $n \in \mathbf{N}$, the Markov property ensures that almost-surely $H(S_n) = \mathbb{E}\{\chi_{\Omega_n}|\mathcal{F}_n\}$. Moreover, we also have for each $m \leq n$

$$\mathbb{E}\{\chi_{\Omega_\infty}|\mathcal{F}_n\} \leq H(S_n) \leq \mathbb{E}\{\chi_{\Omega_m}|\mathcal{F}_n\}.$$

For a fixed $m \in \mathbf{N}$, the martingale convergence theorem ensures that the left side converges almost-surely to $\mathbb{E}\{\chi_{\Omega_\infty}|\mathcal{F}\} = \chi_{\Omega_\infty}$, and that the right side converges almost-surely to $\mathbb{E}\{\chi_{\Omega_m}|\mathcal{F}\} = \chi_{\Omega_m}$ as $n \to +\infty$. Therefore, we have

$$\chi_{\Omega_\infty} \leq \liminf_{n\to+\infty} H(S_n) \leq \limsup_{n\to+\infty} H(S_n) \leq \chi_{\Omega_m},$$

almost-surely for every $m \in \mathbf{N}$. Letting $m \in \mathbf{N}$ tends to $+\infty$, we have almost-surely

$$\lim_{n\to+\infty}\inf_{k\in\mathbf{N}} P^k_{S_n}(\,(\mathscr{S}\setminus A)^k\,) = 1 - \lim_{n\to+\infty} H(S_n) = 1 - \chi_{\Omega_\infty}. \qquad \square$$

Corollary 2.12. *Let $\mathscr{S}$ be a $\star$-compatible and locally compact symmetrization space such that for every $X \in \mathscr{X}$, the set*

$$X^{\mathscr{S}} = \{X^{s_1\dots s_n} : n \in \mathbf{N}, s_1,\dots,s_n \in \mathscr{S}\}$$

has compact closure in $(\mathscr{X}, d)$. Let $\mathcal{A}$ be a $\star$-compatible asymmetry function on $\mathscr{S}$. Assume that there exists $\mathcal{I}^{(\mathscr{S})} \subset \mathscr{S}$ such that $\mathcal{A}$ is strict on $\mathscr{S}\setminus\mathcal{I}^{(\mathscr{S})}$. Let $P : \mathscr{S}\times\mathscr{B}(\mathscr{S}) \to [0,1]$ be a transition function, and $(S_n)_{n\in\mathbf{N}}$ be a time-homogeneous Markov process with transition function P. If

(i) (Stability) the process $(S_n)_{n\in\mathbf{N}}$ is nonevanescent and $\mathcal{I}^{(\mathscr{S})}$ is closed,
(ii) (Continuity) for all $n \in \mathbf{N}$, for every bounded and continuous function $f : \mathscr{S}^n \to \mathbf{R}$, the function $P^n f$ is continuous,
(iii) (Discrimination) for every $s \in \mathscr{S}$, for every nonempty open set $\mathcal{O} \subseteq \mathscr{S}$,

$$\sum_{n\in\mathbf{N}} P^n_s((\mathscr{S}\setminus\mathcal{I}^{(\mathscr{S})})^{n-1}\times\mathcal{O}) > 0,$$

then for every $X \in \mathscr{X}$, the sequence $(X^{S_1\dots S_n})_{n\in\mathbf{N}}$ converges almost-surely to $X^\star$.

Proof. Let us fix $\mathfrak{s}_\star \in \mathscr{S}$. According to proposition 2.9, it is sufficient to check that for every nonempty open set $\mathcal{O} \subseteq \mathscr{S}$ containing $\mathfrak{s}_\star$, we have

$$\mathbb{P}(\{(S_n)_{n\in\mathbf{N}} \text{ enters } \mathcal{O} \text{ infinitely many often}\}) = 1.$$

We thus fix a nonempty open set $\mathcal{O} \subseteq \mathscr{S}$ that contains $\mathfrak{s}_\star$, and we define the function

$$H : \mathscr{S} \mapsto [0,1] : H(s) = 1 - \inf_{k\in\mathbf{N}} P_s^k(\, (\mathscr{S} \setminus \mathcal{O})^k \,).$$

By proposition 2.11, we have almost-surely $\lim_{n\to+\infty} H(S_n) = \chi_{\Omega_\infty}$, where

$$\Omega_\infty = \bigcap_{m\in\mathbf{N}} \bigcup_{i\geq m} \{S_{i+1} \in \mathcal{O}\} = \{(S_n)_{n\in\mathbf{N}} \text{ enters } \mathcal{O} \text{ infinitely many often}\}.$$

Assume by contradiction that there exists $s \in \mathscr{S}$ with $H(s) = 0$, that is: for every $k \in \mathbf{N}$, we have $P_s^k(\, (\mathscr{S} \setminus \mathcal{O})^k \,) = 1$. Since there exists $n \in \mathbf{N}$ such that $P_s^n(\, (\mathscr{S} \setminus \mathcal{I}^{(\mathscr{S})})^{n-1} \times \mathcal{O} \,) > 0$, it follows that

$$0 < P_s^n(\, ((\mathscr{S} \setminus \mathcal{I}^{(\mathscr{S})})^{n-1} \times \mathcal{O}) \cap (\mathscr{S} \setminus \mathcal{O})^n \,) = P_s^n(\emptyset) = 0,$$

which is a contradiction. This proves that the function H is strictly positive. Using Urysohn's lemma, the function H is also lower semi-continuous as supremum of semi-continuous functions. Therefore, for all compact set $K \subseteq \mathscr{S}$, the function H attains a strictly positive minimal value on K, which has to equal 1. Hence, we have proven the essential inclusion

$$\{(S_n)_{n\in\mathbf{N}} \text{ enters } K \text{ infinitely many often}\} \subseteq \Omega_\infty.$$

Since there exists a countable basis of nonempty open sets with compact closure for $\mathscr{S}$, the nonevanescence of $(S_n)_{n\in\mathbf{N}}$ ensures

$$\mathbb{P}(\{(S_n)_{n\in\mathbf{N}} \text{ enters } \mathcal{O} \text{ infinitely many often}\}) = 1. \qquad \square$$

In view of examples, we prove that a strong Feller transition function P always satisfy our continuity assumption for $\{P^n : n \in \mathbf{N}\}$.

Definition 2.13. *Let $\mathscr{S}$ be a topological space. A transition function P on $\mathscr{S}$ is* strong Feller continuous *at $\mathfrak{s}_\star$ if, for every bounded and Borel measurable function $f : \mathscr{S} \to \mathbf{R}$, the map $[s \in \mathscr{S} \mapsto P_s f]$ is continuous at $\mathfrak{s}_\star$.*

Proposition 2.14 ([13, Proposition 6.1.1])**.** *Let $\mathscr{S}$ be a metrizable topological space with countable basis, $\mathfrak{s}_\star \in \mathscr{S}$ and P a transition function on $\mathscr{S}$. If P is strong Feller continuous at $\mathfrak{s}_\star$, then for every $n \in \mathbf{N}$, for every bounded and continuous function $f : \mathscr{S}^n \to \mathbf{N}$, the function $P^n f$ is continuous at $\mathfrak{s}_\star$.*

Proof. If $f : \mathscr{S}^n \to \mathbf{R}$ is bounded and continuous, the function

$$g : \mathscr{S} \to \mathbf{R} : g(x) = P_x^n f(\cdot, x)$$

is bounded and Borel measurable. The measurability follows from the monotone class theorem. Since there holds

$$P_s^{n+1} f = \int_{\mathscr{S}} g(x)\, P_s(\mathrm{d}x),$$

it suffices to prove that for every bounded and Borel measurable function $g : \mathscr{S} \to \mathbf{R}$, Pg is continuous at $\mathfrak{s}_\star$. Let $M > 0$ be a bound for g, and define $h_1 = g + M$. Then h_1 is Borel measurable, positive and bounded. There exists an increasing sequence $(\phi_n)_{n\in\mathbf{N}}$ of simple functions that converges to h_1, and such that the level sets of ϕ_n are disjoint Borel measurable sets. By linearity, the functions $P\phi_n$ are continuous at $\mathfrak{s}_\star$, for every $n \in \mathbf{N}$. Since the sequence $(P\phi_n)_{n\in\mathbf{N}}$ increases to Ph_1 by the monotone convergence theorem, the function Ph_1 is lower semi-continuous, and so is Pg. The same conclusions hold for the function $h_2 = -g + M$, so that Ph_2 is lower semi-continuous, and thus Pg is also upper semi-continuous. $\square$

The converse of proposition 2.14 is false in general. Consider for example a continuous map $\phi : \mathscr{S} \to \mathscr{S}$ which is not trivial, and the transition function

$$P : \mathscr{S} \times \mathscr{B}(\mathscr{S}) \to [0,1] : P_s(A) = \delta_{\phi(s)}(A).$$

The iterated kernels are given by the formula

$$P_s^n f = f(\,\phi(s), \dots, \phi^n(s)\,).$$

The functions $P^n f$ are continuous whenever f is itself continuous and bounded, but the strong Feller continuity may fail in general.

3. Examples

In this section, we give examples for various symmetrizations as Steiner and cap symmetrizations and polarizations. We first recall standard definitions, and then we give examples of application of our abstract method.

3.1. **Various symmetrizations.** We write $\mathscr{H}^k$ the k-dimensional Hausdorff measure in $\mathbf{R}^d$. We work within the metric space $(\mathcal{M}_\sharp(\mathbf{R}^d), d_1)$ of equivalence classes of Borel measurable subsets of $\mathbf{R}^d$ with finite $\mathscr{H}^d$ measure, endowed with the $\mathscr{H}^d$-metric. The space $\mathbf{R}^d$ is equipped with its usual metric.

Spherical nonincreasing rearrangement. The simplest symmetrization transform sets into balls.

Definition 3.1. *Let $A \in \mathscr{B}(\mathbf{R}^d)$ be a Borel measurable set. The spherical nonincreasing rearrangement of A is the open ball $A^\star$ centered on the origin 0, that satisfies $\mathscr{H}^d(A^\star) = \mathscr{H}^d(A)$.*

The induced map on the quotient space $\mathcal{M}_\sharp(\mathbf{R}^d)$ is denoted by $\star$. The fact that $\star$ is an involution is direct. For the nonexpansiveness, one can simply observe that since $\star$ is measure-preserving and monotone, we directly have for all $A, B \in \mathscr{B}(\mathbf{R}^d)$

$$\begin{aligned}
\mathscr{H}^d(A^\star \Delta B^\star) &= \mathscr{H}^d(A^\star \setminus B^\star) + \mathscr{H}^d(B^\star \setminus A^\star)\\
&\le \mathscr{H}^d(A^\star \setminus (B \cap A)^\star) + \mathscr{H}^d(B^\star \setminus (A \cap B)^\star)\\
&= \mathscr{H}^d(A^\star) + \mathscr{H}^d(B^\star) - 2\mathscr{H}^d((B \cap A)^\star)\\
&= \mathscr{H}^d(A) + \mathscr{H}^d(B) - 2\mathscr{H}^d(B \cap A) = \mathscr{H}^d(A \Delta B).
\end{aligned}$$

Steiner symmetrizations. Let be $u \in \mathbf{P}^{d-1}_{\mathbf{R}}$ and $\langle u \rangle$ be its linear span. We write $u^\perp$ the orthogonal complement subspace in $\mathbf{R}^d$. For every $x \in u^\perp$, we write the section

$$A\restriction_x = A \cap (x + \langle u \rangle).$$

Definition 3.2. *Given $u \in \mathbf{P}^{d-1}_{\mathbf{R}}$ and a Borel measurable set $A \in \mathscr{B}(\mathbf{R}^d)$, the Steiner symmetrization of A with respect to u is the unique $A^u \in \mathscr{B}(\mathbf{R}^d)$ such that*

- *for every $x \in u^\perp$, $A^u\restriction_x$ is an open ball of $(x + \langle u \rangle)$,*
- *for every $x \in u^\perp$, $A^u\restriction_x$ is centered on 0,*
- *for every $x \in u^\perp$, $\mathscr{H}^1(A^u\restriction_x) = \mathscr{H}^1(A\restriction_x)$.*

The induced maps on the quotient $\mathcal{M}_\sharp(\mathbf{R}^d)$ are the usual Steiner symmetrizations. The set $\mathcal{S}$ of Steiner symmetrizations being in one-to-one correspondence with $\mathbf{P}^{d-1}_{\mathbf{R}}$, we equip it with the induced topology of $\mathbf{P}^{d-1}_{\mathbf{R}}$, so that $\mathcal{S}$ is a metrizable and compact topological space with countable basis.

Polarizations. Fix $\hat{e} \in S^{d-1}$ and $r \ge 0$. The corresponding affine half-subspace is defined by

$$H^{\hat{e},r} = \{x \in \mathbf{R}^d : x \cdot \hat{e} \le r\}.$$

There is a unique nontrivial reflection $\sigma^{\hat{e},r}$ of $\mathbf{R}^d$ that leaves the boundary of $H^{\hat{e},r}$ invariant.

Definition 3.3. *Given $\hat{e} \in S^{d-1}$, $r \geq 0$ and a Borel measurable set $A \in \mathscr{B}(\mathbf{R}^d)$, the polarization of A is defined as the unique $A^{\hat{e},r} \in \mathscr{B}(\mathbf{R}^d)$ satisfying the following axioms:*

– *if $x \in H^{\hat{e},r}$, then $x \in A^{\hat{e},r}$ if, and only if, $x \in H^{\hat{e},r} \cup \sigma^{\hat{e},r}(H^{\hat{e},r})$,*
– *if $x \notin H^{\hat{e},r}$, then $x \in A^{\hat{e},r}$ if, and only if, $x \in H^{\hat{e},r} \cap \sigma^{\hat{e},r}(H^{\hat{e},r})$.*

The induced maps on the quotient $\mathcal{M}_\sharp(\mathbf{R}^d)$ are the usual polarizations. The set $\mathcal{H}$ of polarizations is in one-to-one correspondence with $S^{d-1} \times [0, +\infty)$. We equip it with the induced topology of $S^{d-1} \times [0, +\infty)$, so that $\mathcal{H}$ is a metrizable and locally compact topological space with countable basis. Through this identification, we define the closed subset

$$\mathcal{I}^{(\mathcal{H})} = S^{d-1} \times \{0\}.$$

It corresponds to affine half-subspaces that contains 0 in their usual boundary.

Cap symmetrizations. Fix $\hat{e} \in S^{d-1}$ and $r \geq 0$. For $t \geq 0$, define the spherical section of a set $A \subseteq \mathbf{R}^d$ as

$$A\restriction_t = A \cap \partial B(r\hat{e}, t).$$

Definition 3.4. *Given $r \geq 0$ and a Borel measurable set $A \in \mathscr{B}(\mathbf{R}^d)$, the cap symmetrization of A with respect to $(\hat{e}, r)$ is the unique $A^{\hat{e},r} \in \mathscr{B}(\mathbf{R}^d)$ that satisfies the following axiom:*

– *for every $t \geq 0$, $A^{\hat{e},r}\restriction_t$ is an open ball of $\partial B(r\hat{e}, t)$,*
– *for every $x \in u^\perp$, $A^{\hat{e},r}\restriction_t$ is centered on $(r - t)\hat{e}$*
– *for every $t \geq 0$, $\mathscr{H}^{d-1}(A^{\hat{e},r}\restriction_t) = \mathscr{H}^{d-1}(A\restriction_t)$.*

The induced maps on the quotient $\mathcal{M}_\sharp(\mathbf{R}^d)$ form the usual set of cap symmetrizations. The set $\mathcal{L}$ of cap symmetrizations being in one-to-one correspondence with $S^{d-1} \times [0, +\infty)$, we equip it with the induced topology of $S^{d-1} \times [0, +\infty)$, so that $\mathcal{L}$ is a metrizable and locally compact topological space with countable basis. We define the closed subset

$$\mathcal{I}^{(\mathcal{L})} = S^{d-1} \times \{0\}.$$

It corresponds to cap symmetrizations with respect to half-lines whose initial points are the origin 0.

Common properties of the examples of symmetrizations. Steiner symmetrizations, cap symmetrizations and polarizations enjoy important common properties, which we recall in the next proposition. The following result being classical in the field of symmetrizations, we omit the proof.

Proposition 3.5. *The sets $\mathcal{S}, \mathcal{H}$ and $\mathcal{L}$ acting on $(\mathcal{M}_\sharp(\mathbf{R}^d), d_1)$ are $\star$-compatible symmetrization spaces and for every $X \in \mathcal{M}_\sharp(\mathbf{R}^d)$, the sets*

$$X^{\mathcal{S}} = \{X^{s_1 \dots s_n} : n \in \mathbf{N}, s_1, \dots, s_n \in \mathcal{S}\},$$
$$X^{\mathcal{H}} = \{X^{s_1 \dots s_n} : n \in \mathbf{N}, s_1, \dots, s_n \in \mathcal{H}\},$$
$$X^{\mathcal{L}} = \{X^{s_1 \dots s_n} : n \in \mathbf{N}, s_1, \dots, s_n \in \mathcal{L}\}$$

have compact closure in $(\mathcal{M}_\sharp(\mathbf{R}^d), d_1)$. The function

$$\mathcal{A} : \mathcal{M}_\sharp(\mathbf{R}^d) \to \mathbf{R}^+ : \mathcal{A}(X) = \int_X \frac{|x|^2}{1 + |x|^2} \, \mathrm{d}x$$

is a $\star$-compatible asymmetry function on $\mathcal{S}$, on $\mathcal{H}$ and on $\mathcal{L}$. Moreover, $\mathcal{A}$ is a strict asymmetry function on $\mathcal{S}$, on $\mathcal{H} \setminus \mathcal{I}^{(\mathcal{H})}$ and on $\mathcal{L} \setminus \mathcal{I}^{(\mathcal{L})}$.

The previous proposition is straightforward for polarizations [6; 7, Polarization identity]. The compactness property follows from the Kolmogorov-Riesz compactness theorem. Once one has the result for polarizations, one can extend it to Steiner and cap symmetrizations by an approximation argument [22].

3.2. **Markov Steiner symmetrizations.** In view of the discussion above, we can prove directly theorem 1.2.

Proof of theorem 1.2. The function

$$\mathcal{A} : \mathcal{M}_\sharp(\mathbf{R}^d) \to \mathbf{R}^+ : \mathcal{A}(X) = \int\limits_X \frac{|x|^2}{1+|x|^2}\, dx$$

is a strict asymmetry function on $\mathcal{S}$, according to proposition 3.5. The fact that the continuity condition (ii) in theorem 1.2 is equivalent to the continuity condition (ii) of proposition 2.9 is a consequence of Urysohn's lemma. We can apply proposition 2.9 to get the desired result. □

Example 3.6 (Random walk in $\mathbf{P}^{d-1}_{\mathbf{R}}$)**.** We fix $r > 0$ and we write σ for the Haar measure on $\mathbf{P}^{d-1}_{\mathbf{R}}$. The map $[e \in \mathbf{P}^{d-1}_{\mathbf{R}} \mapsto \sigma(B(e,r))]$ is constant. We fix $e \in \mathbf{P}^{d-1}_{\mathbf{R}}$ and we define the transition function

$$P : \mathbf{P}^{d-1}_{\mathbf{R}} \times \mathscr{B}(\mathbf{P}^{d-1}_{\mathbf{R}}) \to [0,1] : P_s(A) = \int\limits_{B(s,r)} \chi_A(x)\, \frac{\sigma(\mathrm{d}x)}{\sigma(B(e,r))}.$$

The transition function P is strong Feller continuous everywhere on $\mathbf{P}^{d-1}_{\mathbf{R}}$. According to proposition 2.14, the family $\{P^n : n \in \mathbf{N}\}$ enjoys the usual continuity assumption at every point. We also have, for every nonempty open set $\mathcal{O} \subseteq \mathbf{P}^{d-1}_{\mathbf{R}}$ and for every $s \in \mathbf{P}^{d-1}_{\mathbf{R}}$,

$$\sum_{n\in\mathbf{N}} P^n_s(\, (\mathbf{P}^{d-1}_{\mathbf{R}})^{n-1} \times \mathcal{O}\,) > 0.$$

This last inequality can be proven by noting that, for every $n \in \mathbf{N}$, for every $s \in \mathbf{P}^{d-1}_{\mathbf{R}}$, the probability measure

$$H^n_s : \mathscr{B}(\mathbf{P}^{d-1}_{\mathbf{R}}) \to [0,1] : H^n_s(A) = P^n_s(\, (\mathbf{P}^{d-1}_{\mathbf{R}})^{n-1} \times A\,),$$

has $\overline{B(s,nr)}$ as support. Since $\mathbf{P}^{d-1}_{\mathbf{R}}$ is compact, any Markov process on $\mathbf{P}^{d-1}_{\mathbf{R}}$ is nonevanescent. Corollary 2.12 ensures that any time-homogeneous Markov process $(S_n)_{n\in\mathbf{N}}$ with transition function P satisfies that for every $X \in \mathcal{M}_\sharp(\mathbf{R}^d)$, the sequence $(X^{S_1\cdots S_n})_{n\in\mathbf{N}}$ converges almost-surely to $X^\star$.

Proposition 3.7 (Deterministic Steiner symmetrizations)**.** *Let $\phi : \mathbf{P}^{d-1}_{\mathbf{R}} \to \mathbf{P}^{d-1}_{\mathbf{R}}$ be a continuous map and $s \in \mathbf{P}^{d-1}_{\mathbf{R}}$ such that*

$$\overline{\{\phi^n(s) : n \in \mathbf{N}\}} = \mathbf{P}^{d-1}_{\mathbf{R}}.$$

For every $X \in \mathcal{M}_\sharp(\mathbf{R}^d)$, the sequence $(X^{\phi(s)\cdots\phi^n(s)})_{n\in\mathbf{N}}$ converges in measure to $X^\star$.

Proof. Let us define for all $n \in \mathbf{N}$, $S_n = \phi^n(S)$. The sequence $(S_n)_{n\in\mathbf{N}}$ is a time-homogeneous Markov process on $\mathbf{P}^{d-1}_{\mathbf{R}}$. The iterated kernels are given for every $n \in \mathbf{N}$ by the formula

$$P^n_s f = f(\,\phi(s), \dots, \phi^n(s)\,).$$

The recurrence condition (i) and the discrimination condition (iii) of proposition 2.9 both follow from the assumption that the orbit of s under ϕ is dense in $\mathbf{P}^{d-1}_{\mathbf{R}}$. Since ϕ is continuous, the continuity condition (ii) is satisfied whenever f is continuous and bounded. We can thus apply proposition 2.9 with limit point $\mathfrak{s}_\star = s$. The proof is done. □

The previous proposition can be thought of as a generalization of [2, Theorem 5.1], where the authors studied Kronecker sequence of deterministic Steiner symmetrizations of the form $(\,(\cos(n\alpha), \sin(n\alpha))\,)_{n\in\mathbf{N}}$, with α/π irrational. Their analysis is based on the convergence in shape. Our result can be applied for other deterministic sequence, and has straightforward generalizations for cap symmetrizations and polarizations.

Counterexample. The continuity condition (ii) in theorem 1.2 is necessary. Let us consider a sequence $(\alpha_n)_{n\in\mathbf{N}}$ in $\mathbf{P}^{d-1}_{\mathbf{R}}$ such that $\alpha_i \neq \alpha_j$ for $i \neq j$, and such that the set $\{\alpha_n : n \in \mathbf{N}\}$ is dense in $\mathbf{P}^{d-1}_{\mathbf{R}}$. We define the transition function on $\mathbf{P}^{d-1}_{\mathbf{R}}$ through

$$P_s = \begin{cases} \delta_{\alpha_{n+1}} & \text{if } s = \alpha_n \text{ for some } n \in \mathbf{N}, \\ \delta_{\alpha_1} & \text{otherwise} \end{cases}.$$

If $(S_n)_{n\in\mathbf{N}}$ is a time-homogeneous Markov process associated with the transition function P, then almost-surely $S_n = \alpha_n$ for every $n \in \mathbf{N}$. Hence, it is a straightforward computation to check that the hypothesis of theorem 1.2 are satisfied with $\mathfrak{s}_\star = \alpha_1$, except that we miss the continuity property. Actually, in general, there exists $X \in \mathcal{M}_\sharp(\mathbf{R}^d)$ such that $(X^{S_1\cdots S_n})_{n\in\mathbf{N}}$ fails to convergence to $X^\star$ almost-surely [3, 24].

In this example, note that one could endow $\mathbf{P}^{d-1}_{\mathbf{R}}$ with the discrete topology. The continuity assumption is then trivial, but the recurrence condition forces the process to have a finite cycle. This is the situation of iterated Steiner symmetrizations using a finite number of directions [10].

3.3. Markov cap symmetrizations and polarizations on $(0, +\infty)$.

Proof of theorem 1.3. Since we identify $\mathcal{L}$ and $\mathcal{H}$ with $S^{d-1} \times [0, +\infty)$, let us recall that $\mathcal{I}^{(\mathcal{L})} = S^{d-1} \times \{0\} = \mathcal{I}^{(\mathcal{H})}$. We consider the usual asymmetry function

$$\mathcal{A} : \mathcal{M}_\sharp(\mathbf{R}^d) \to \mathbf{R}^+ : \mathcal{A}(X) = \int_X \frac{|x|^2}{1 + |x|^2}\, dx.$$

It is a $\star$-compatible asymmetry function on $\mathcal{L}$ and $\mathcal{H}$ which is strict on the subsets $\mathcal{L} \setminus \mathcal{I}^{(\mathcal{L})}$ and $\mathcal{H} \setminus \mathcal{I}^{(\mathcal{H})}$ (proposition 3.5). In these settings, the result follows from proposition 2.9. $\square$

For polarizations, the following elementary result shows that the nonevanescence assumption in corollary 2.12 is actually required for the sequence to approximate the spherical nonincreasing rearrangement. This property fails in general for cap symmetrizations.

Proposition 3.8. *Let $(s_n)_{n\in\mathbf{N}}$ be a sequence of polarizations written for every $n \in \mathbf{N}$ as $s_n = (\hat{e}_n, r_n)$ with $e_n \in S^{d-1}$ and $r_n \in [0, +\infty)$. If $\liminf_{n\to+\infty} r_n > 0$, then there exists $X \in \mathcal{M}_\sharp(\mathbf{R}^d)$ of finite measure such that the sequence $(X^{s_1\cdots s_n})_{n\in\mathbf{N}}$ does not converge.*

Proof. The condition $\liminf_{n\to+\infty} r_n > 0$ shows that there exists $N \in \mathbf{N}$ such that, for every $n \geq N$, we have $r_n \geq \delta$, for some $\delta > 0$. We define

$$r = \min\{\delta, \min\{r_i : i \in \{1, \dots, N-1\}, r_i > 0\}\},$$

and $X \in \mathcal{M}_\sharp(\mathbf{R}^d)$ the (equivalent class of the) annulus

$$X = B(0, r) \setminus B(0, r/2).$$

Then clearly $X \neq X^\star$. For $n \in \mathbf{N}$, two cases may occur. If $r_n \geq r$, then X is contained in the half-space $(\hat{e}_n, r_n)$, and thus $X = X^{(\hat{e}_n, r_n)}$. If $r_n < r$, then we must have by construction $r_n = 0$. Since X is radially symmetric, we have $X = X^{(\hat{e}_n, 0)}$, so that X remains fixed by all polarizations of the set $\{s_n : n \in \mathbf{N}\}$. $\square$

Example 3.9 (Brownian motion on $(0, +\infty)$)**.** Let $(Z_n)_{n\in\mathbf{N}}$ be a sequence of independent and identically distributed variables on $(0, +\infty)$ with density function ρ, and define for every $n \in \mathbf{N}$

$$\begin{cases} W_1 & = Z_1, \\ W_{n+1} & = Z_{n+1} \cdot W_n. \end{cases}$$

Let $(U_n)_{n\in\mathbf{N}}$ be a sequence of independent and identically distributed variables on S^{d-1} with distribution μ. We assume that $\operatorname{supp}(\mu) \times \operatorname{supp}(\rho) = S^{d-1} \times (0, +\infty)$ and

$$\int_{(0,+\infty)} s\rho(s)\,\mathrm{d}s = 1.$$

The transition function of the time-homogeneous Markov process $(W_n)_{n\in\mathbf{N}}$ is given by

$$P : \mathscr{B}(\,(0,+\infty)\,) \times (0,+\infty) \to [0,1] : P_x(A) = \int\limits_{(0,+\infty)} \chi_A(sx)\rho(s)\,\mathrm{d}s.$$

For every continuous and bounded function $f : (0,+\infty) \to \mathbf{R}$, we have

$$P_x f = \int\limits_{(0,+\infty)} f(xs)\rho(s)\,\mathrm{d}s,$$

which is continuous with respect to the parameter $x \in (0,+\infty)$. By assumption on the support of ρ, every nonempty open set of $(0,+\infty)$ is reachable with positive probability from every point $x \in (0,+\infty)$, in one step. We now study the nonevanescence of $(W_n)_{n\in\mathbf{N}}$. It remains to prove that the process $((U_n, W_n))_{n\in\mathbf{N}}$ is nonevanescent. Since S^{d-1} is compact, it suffices to shows that $(W_n)_{n\in\mathbf{N}}$ is nonevanescent in $(0,+\infty)$. We proceed by following the classical drift criterion [13, chapters 8, 9]. By assumption, we have

$$\int\limits_{(0,+\infty)} s\, P_x(\mathrm{d}s) = x \int\limits_{(0,+\infty)} s\rho(s)\,\mathrm{d}s = x.$$

We write $E = \{\exists K \subseteq \mathscr{S} : K \text{ compact}, (W_n)_{n\in\mathbf{N}} \text{ enters } K \text{ infinitely many often}\} \in \mathcal{F}$. Assume by contradiction that $\mathbb{P}(E) < 1$, and let $K \subseteq (0,+\infty)$ be a compact set. Then there exists $k \in \mathbf{N}$ such that $0 < \mathbb{P}(\{\forall i > k : W_i \notin K\} \setminus E)$. Denoting by μ the initial distribution of the process, the distribution of W_k is given by the measure

$$\mu P^k : \mathscr{B}(\mathscr{S}) \to [0,1] : \mu P^k(A) = \int\limits_{\mathscr{S}} P_s^k(\mathscr{S}^{k-1} \times A)\,\mu(\mathrm{d}s).$$

Hence, the new process $(G_n)_{n\in\mathbf{N}}$ defined for every $n \in \mathbf{N}$ by $G_n = W_{n+k-1}$ is a time-homogeneous Markov process with transition function P, and initial distribution μP^k. The random variable

$$T = \min\{n \in \mathbf{N} : G_n \in K\},$$

is adapted to the filtration $(\mathcal{F}_{n+k-1})_{n\in\mathbf{N}}$ and it satisfies $\{T = +\infty\} = \{\forall i \geq k : W_i \notin K\}$. We use the martingale convergence theorem [4, Theorem 35.5] to show that the set $(\{T = +\infty\} \setminus E) \in \mathcal{F}$ has null $\mathbb{P}$-measure [13, Proposition 9.4.1]. To see this, observe that the stochastic process $(M_n)_{n\in\mathbf{N}}$ defined for every $n \in \mathbf{N}$ by

$$M_n = G_n \chi_{\{T\geq n\}},$$

is a positive martingale with respect to the filtration $(\mathcal{F}_{n+k-1})_{n\in\mathbf{N}}$, by construction of c. By the martingale convergence theorem, it converges $\mathbb{P}$-almost-surely to some $\mathcal{F}$-measurable random variable M_∞, which is $\mathbb{P}$-almost-surely finite. Therefore, we have $\mathbb{P}$-almost-surely

$$\chi_{\{T=+\infty\}} M_\infty = \chi_{\{T=+\infty\}} \lim_{n\to+\infty} W_{n+k-1},$$

so that $\{T = +\infty\} \setminus E$ has null $\mathbb{P}$-measure, which contradicts the construction of $k \in \mathbf{N}$. Therefore, the process $(W_n)_{n\in\mathbf{N}}$ is nonevanescent, and so is $((U_n, W_n))_{n\in\mathbf{N}}$. We deduce from corollary 2.12 that for every $X \in \mathcal{M}_\sharp(\mathbf{R}^d)$, the sequence of successive cap symmetrizations (resp. polarizations) $(X^{(U_1,W_1)\dots(U_n,W_n)})_{n\in\mathbf{N}}$ converges in measure to $X^\star$.

3.4. **Example for Markov cap symmetrizations on** $[0,+\infty)$**.** The previous examples concern random walks on S^{d-1} or $S^{d-1}\times(0,+\infty)$, viewed as symmetrization spaces equipped with a strict asymmetry function. In this section, we present a *new example* of cap symmetrization space $\mathcal{L}^\sharp \subset \mathcal{L}$. This new symmetrization space satisfies that the subset

$$\overline{\mathcal{L}^\sharp \setminus \mathcal{I}^{(\mathcal{L})}} \Subset \mathcal{L}^\sharp$$

is *not* a symmetrization space. In other words, our example occurs in a subset $\mathcal{L}^\sharp$ where the nonstrict symmetrization of $\mathcal{I}^{(\mathcal{L})}$ are needed for the convergence. To our knowledge, such symmetrization spaces are unknown from the literature. We will see that it is possible to construct a continuous and nonevanescent time-homogeneous Markov process in $\mathcal{L}^\sharp$ such that the universal convergence fails. This will illustrate the necessity of the discrimination condition (iii) in proposition 2.9.

As a preparation for the results of this paragraph, we first recall a known model [13] of random walk in $[0,+\infty)$. Although the assumptions in the following lemma can be weakened, we keep them as simple as possible to make the analysis easy.

Lemma 3.10 (Brownian motion on $[0,+\infty)$)**.** *Let $(Z_n)_{n\in\mathbf{N}}$ be a sequence of independent and identically distributed variables on $\mathbf{R}$ with density function ρ, and define for every $n\in\mathbf{N}$*

$$\begin{cases} W_1 &= Z_1,\\ W_{n+1} &= \max\{W_n + Z_{n+1}, 0\}.\end{cases}$$

If $\operatorname{supp}(\rho)=\mathbf{R}$ and if there exists $\delta>0$ such that

$$\int\limits_{(0,+\infty)} s\rho(s)\,\mathrm{d}s < 0 < \inf_{[-\delta,\delta]}\rho,$$

then $(W_n)_{n\in\mathbf{N}}$ is nonevanescent time-homogeneous Markov process with strong Feller continuous transition function P such that for every $s\in[0,+\infty)$, for every nonempty open set $\mathcal{O}\subseteq[0,+\infty)$, we have

$$\sum_{n\in\mathbf{N}} P^n_s(\,(0,+\infty)^{n-1}\times\mathcal{O}\,)>0.$$

Proof. The process $(W_n)_{n\in\mathbf{N}}$ is a Markov process, whose transition function is given by

$$P:\mathscr{B}(\,[0,+\infty)\,)\times[0,+\infty)\to[0,1]: P_s(A)=\Gamma(A\setminus\{0\}-s)+\Gamma(\,(-\infty,-s]\,)\delta_0(A).$$

Here Γ stands for the Lebesgue measure on $\mathbf{R}$ weighted by ρ. The continuity of translations [25, Lemma 4.3.8] and the dominated convergence theorem that P is strong Feller continuous. The nonevanscence of $(W_n)_{n\in\mathbf{N}}$ is known [13, Proposition 9.4.5, Theorem 9.4.1] and the proof is similar to the proof given for the brownian motion on $(0,+\infty)$. We omit the details.

Let us prove that for every $s\in[0,+\infty)$, for every nonempty open set $\mathcal{O}\subseteq[0,+\infty)$, we have

$$\sum_{n\in\mathbf{N}} P^n_s(\,(0,+\infty)^{n-1}\times\mathcal{O}\,)>0.$$

First observe that it is sufficient to prove the claim for every nonempty open set of $(0,+\infty)$. By assumption on ρ, there exists $C>0$ such that for every Borel measurable set $A\subseteq[0,+\infty)$,

$$\int\limits_A \rho(x)\,\mathrm{d}x \geq C\int\limits_A \chi_{[-\delta,\delta]}\,\mathrm{d}x.$$

Therefore, by comparison of series, we can assume without loss of generality that the distribution Γ is the uniform distribution on the interval $[-\delta,\delta]$. A straightforward computation then shows that for every $s\in[0,+\infty)$ and for every $n\in\mathbf{N}$, the measure

$$H^n_s:\mathscr{B}(\,(0,+\infty)\,)\to[0,1]: H^n_s(A)=P^n_s(\,(0,+\infty)^{n-1}\times A\,)$$

has support $[\max\{0,s-n\delta\},s+n\delta]$, which concludes the proof. $\square$

Definition 3.11. *Fix $\hat{e}\in S^{d-1}$. The truncated cap symmetrization space $\mathcal{L}^\sharp\subseteq\mathcal{L}$ is defined from $\{-1,1\}\times[0,+\infty)$ through the action*

$$\begin{cases} (1,r)\mapsto \textit{the cap symmetrization } (\hat{e},r)\\ (-1,r)\mapsto \textit{the cap symmetrization } (-\hat{e},0).\end{cases}$$

Observe that $\mathcal{L}^\sharp$ posses two connected components, one of them being $\{-1\}\times[0,+\infty)$ whose elements act the same way on $\mathcal{M}_\sharp(\mathbf{R}^d)$.

Proposition 3.12. *The symmetrization space $\mathcal{L}^\sharp$ acting on $(\mathcal{M}_\sharp(\mathbf{R}^d), d)$ as in definition 3.11, is $\star$-compatible.*

Proof. The algebraic relation $s\circ\star=\star=\star\circ s$ is valid for every $s\in\mathcal{L}$, so it is true in $\mathcal{L}^\sharp$. Let $A\in\mathscr{B}(\mathbf{R}^d)$ be a Borel measurable set such that $\mathscr{H}^d(A^s\Delta A)=0$ for every $s\in\mathcal{L}^\sharp$. Assume by contradiction that $\mathscr{H}^d(A^\star\Delta A)>0$. Consider the Borel measurable set

$$B=A^{(-\hat{e},0)(\hat{e},0)}.$$

Since $A=B$ in $\mathcal{M}_\sharp(\mathbf{R}^d)$, we clearly have $B^\star=A^\star$, $\mathscr{H}^d(B^\star\Delta B)>0$ and $\mathscr{H}^d(B^s\Delta B)=0$ for every $s\in\mathcal{L}^\sharp$. Moreover, the set $B\cap\partial B(0,t)$ equals $\emptyset$ or $\partial B(0,t)$, for every $t>0$, by construction of cap symmetrizations and by assumption on A. Since $\mathscr{H}^d(B)=\mathscr{H}^d(B^\star)$, the sets $B\setminus B^\star$ and $B^\star\setminus B$ share the same positive Hausdorff measure. There exists $x_1=(\hat{e},r_1)\in B^\star\setminus B$ and $x_2=(\hat{e},r_2)\in B\setminus B^\star$, with $r_1,r_2>0$, and such that for every $r>0$, we have

$$\mathscr{H}^d(\,B(x_1,r)\cap(B^\star\setminus B)\,)>0,\quad \mathscr{H}^d(\,B(x_2,r)\cap(B\setminus B^\star)\,)>0.$$

But then we get $\mathscr{H}^d(B^{(\hat{e},\frac{r_1+r_2}{2})}\Delta B)>0$, which contradicts the construction of B. Therefore, we should have $\mathscr{H}^d(B\Delta B^\star)=0$. ☐

Example 3.13 (Random walk on $\mathcal{L}^\sharp$)**.** Consider a random walk $(W_n)_{n\in\mathbf{N}}$ on $[0,+\infty)$ constructed as in lemma 3.10. We write

$$K:[0,+\infty)\times\mathscr{B}(\,[0,+\infty)\,)\to[0,1]$$

the transition function of $(W_n)_{n\in\mathbf{N}}$. Consider a Markov process $(U_n)_{n\in\mathbf{N}}$ on $\{-1,1\}$, independent of $(W_n)_{n\in\mathbf{N}}$, and whose transition function D has the form

$$H_i(A)=\beta_i\delta_{\{i\}}(A)+(1-\beta_i)\delta_{\{-i\}}(A),$$

where $\beta_1,\beta_{-1}\in(0,1)$. The product process $((U_n,W_n))_{n\in\mathbf{N}}$ is a Markov process on $\{-1,1\}\times[0,+\infty)$, which satisfies the following conditions:

- the transition function P of the product process satisfies

  $$P_{(i,r)}(A)=H_i(\{1\})K_r(\pi^+(A))+H_i(\{-1\})K_r(\pi^-(A)),$$

  where $\pi^-(A),\pi^+(A)\in\mathscr{B}(\,[0,+\infty)\,)$ are the unique Borel measurable sets that satisfies

  $$A\cap\{-1\}\times[0,+\infty)=\{-1\}\times\pi^-(A),$$

  and

  $$A\cap\{1\}\times[0,+\infty)=\{1\}\times\pi^+(A).$$

- P is strong Feller continuous and for every nonempty open set $\mathcal{O}\subseteq\{-1,1\}\times[0,+\infty)$, we have for every $(i,r)\in\{-1,1\}\times[0,+\infty)$

  $$\sum_{n\in\mathbf{N}}P^n_{(i,r)}(\,(\{-1,1\}\times(0,+\infty))^{n-1}\times\mathcal{O}\,)>0.$$

- the process $((U_n,W_n))_{n\in\mathbf{N}}$ is nonevanescent, since so is $(W_n)_{n\in\mathbf{N}}$.

According to corollary 2.12, for every $X\in\mathcal{M}_\sharp(\mathbf{R}^d)$, the sequence of successive symmetrizations $(X^{(U_1,W_1)\dots(U_n,W_n)})_{n\in\mathbf{N}}$ converges almost-surely in measure to $X^\star$.

Counterexample (Convergence failure for Markov process on $\mathcal{L}^\sharp$)**.** We illustrate the necessity of the discrimination condition (iii) in proposition 2.9. We define the transition function

$$P:\mathcal{L}^\sharp\times\mathscr{B}(\mathcal{L}^\sharp)\to[0,1]$$

through the formula

$$P_{(\hat{e},r)}(A)=\frac{1}{2}(\delta_{\hat{e}}\otimes\mu+\delta_{-\hat{e}}\otimes\delta_0)(A),\qquad P_{(-\hat{e},r)}(A)=\frac{1}{2}(\delta_{-\hat{e}}\otimes\mu+\delta_{\hat{e}}\otimes\delta_0)(A).$$

where $\mu : \mathscr{B}([0,+\infty)) \to [0,1]$ is a strictly positive distribution. For example, one could choose μ given by a half normal distribution

$$\mu(A) = \sqrt{\frac{2}{\pi}} \int_A e^{-\frac{x^2}{2}} \,\mathrm{d}x.$$

The continuity and discrimination properties are easily checked. The definition of P forces the process to go on the boundary before to jump on the other half of $\mathcal{L}^\sharp$. Considering half-disks

$$D^+ = \{(z_1, z_2) \in \mathbf{R}^2 : {z_1}^2 + {z_2}^2 < 1, z_2 > 0\},$$
$$D^- = \{(z_1, z_2) \in \mathbf{R}^2 : {z_1}^2 + {z_2}^2 < 1, z_2 < 0\},$$

it is a straightforward computation to see that the sequences of successive cap symmetrizations $((D^+)^{S_1 \cdots S_n})_{n \in \mathbf{N}}$ and $((D^-)^{S_1 \cdots S_n})_{n \in \mathbf{N}}$ alternate between D^+ and D^-, and the almost-sure convergence does not occur.

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